5. POLYGONS

§5.1. Polygons

A **polygon** is a finite sequence of at least three distinct points in the Euclidean plane. I shall extend the use of the symbol Δ from triangles to polygons in general. So I will denote the polygon with vertices P_1, P_2, …, P_n by $\mathscr{P}(\mathbf{P_1P_2 \ldots P_n})$ provided the vertices are in the cyclic order (clockwise or anticlockwise) as listed. If it has *n* vertices it is called an ***n*-gon**.

The name 'polygon' comes from the Greek 'poly' meaning 'many' and 'gon' meaning angles.

Special names for various values of n are:

n	**name**
3	triangle (or trigon)
4	quadrangle
5	pentagon
6	hexagon
7	septagon
8	octagon
9	nonagon
10	decagon

The **edges** of the polygon $\mathscr{P}(P_1P_2, \ldots P_n)$ are the line segments P_1P_2, P_2P_3, …, $P_{n-1}P_n$ and P_nP_1. The (**interior**) **angles** are $\angle P_1P_2P_3$, …, $\angle P_{n-2}P_{n-1}P_n$, $\angle P_{n-1}P_nP_1$ and $\angle P_nP_1P_2$.

Two vertices P_i and P_j are **adjacent** if P_iP_j is an edge. Two edges are **adjacent** if they share a common vertex. A **diagonal** is a line segment P_iP_j which is not an edge.

A polygon is **convex** if all the diagonals lie within the region enclosed by the edges. It is **simple** if no edge intersects any other.

A **regular polygon** is one where all sides have equal length and all angles are equal. A regular triangle is equilateral and a regular quadrangle is a square.

The angles of a triangle total 180°. For each extra side we add 180° to this total.

Theorem1:

The interior angles of an n-gon total $180(n - 2)°$

Proof: Let $\alpha(P)$ denote the sum of the interior angles of the polygon P. I prove this theorem by induction on n.

If $n = 3$, this is the 'angles of a triangle' theorem which was proved earlier.

Suppose $n > 3$ and suppose that the theorem holds for n.

Consider a polygon $\mathcal{P}(P_1P_2, \ldots P_nP_{n+1})$. Cut off the triangle $\Delta P_nP_{n+1}P_1$.

What remains is the n-gon $\mathcal{P}(P_1P_2, \ldots P_n)$.

$$\begin{aligned}\therefore \alpha(\mathcal{P}(P_1 \ldots P_nP_{n+1})) &= \alpha(\mathcal{P}(P_1 \ldots P_n)) + \alpha(\Delta P_nP_{n+1} P_1)\\ &= 180(n - 2)° + 180°\\ &= 180(n + 1 - 2)°,\end{aligned}$$

So the theorem holds for $n + 1$. Therefore, by induction, it holds for all $n \geq 3$.

§5.2. Regular Polygons

A polygon $\mathcal{P}(P_1P_2 \ldots P_n)$ is **regular** if all sides have the same length and all angles are equal. The angles of a regular n-gon are all of size $\dfrac{180(n - 2)}{n}°$.

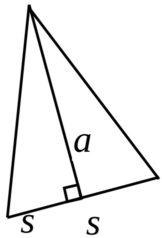
Special names for $n = 3$ and $n = 4$ are equilateral triangle and square respectively.

Clearly all the perpendicular bisectors of the sides of a regular polygon $\mathcal{P}$ are concurrent and the point of concurrence is called the **centre** of $\mathcal{P}$. The perpendicular distance of the centre from each side of $\mathcal{P}$ is clearly equal and this distance is called the apothem of $\mathcal{P}$.

Theorem 2: The apothem of a regular *n*-gon of side *s* is

$$\frac{s}{2\tan\left(\frac{180}{n}\right)}.$$

Proof: Let the apothem be *a*.

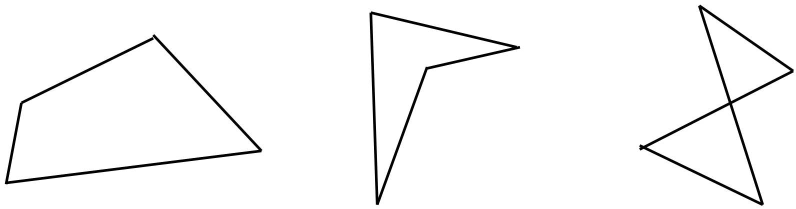


Then $\tan\left(\frac{360}{2n}\right) = \frac{s/2}{a}$, so a = $\frac{s}{2\tan\left(\frac{180}{n}\right)}$.

Theorem 3: The area of a regular *n*-gon ½ P*a* where P is the perimeter and a is the apothem.
Proof: The area of the piece illustrated above is ½ *as*. Hence the total area = ½ *asn* = ½ P*a*.

§5.3. Quadrangles

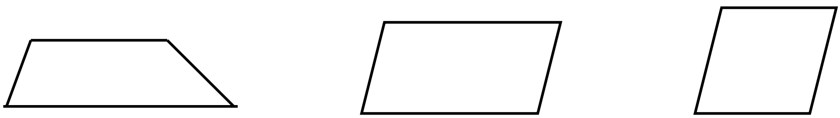
A **quadrangle** is a polygon with four vertices (and four sides). There are three possibilities for a quadrangle. It can be convex and simple, non-convex and simple or non-convex and non-simple.



convex and simple **non-convex and simple** **non-simple and non-simple**

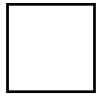
A **quadrilateral** is a quadrangle that is a simple quadrangle. It can be either convex or non-convex. We denote a quadrilateral by its four vertices in order, either clockwise or anticlockwise. So if $\mathcal{P}$(ABCD) is a quadrilateral then the sides are AB, BC, CD and DA. The **diagonals** are AC and BD. Since quadrilaterals are convex these lie within the region enclosed by the sides.

There are special names for certain special quadrilaterals. A **trapezium** is a quadrilateral with a pair of parallel sides. A **parallelogram** is a quadrilateral with two pairs of parallel sides. We have shown that a quadrilateral is a parallelogram if and only if it has one pair of parallel sides of equal length. A **rhombus** is a parallelogram with four equal sides. A **rectangle** is a parallelogram where all the angles are right angles. A **square** is a rhombus which is also a rectangle.



trapezium **parallelogram** **rhombus**

rectangle



square

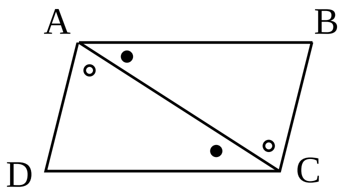
The angles of a quadrilateral total 180(4 – 2)° = 360°.

§5.4. Parallelograms

Theorem 5: (1) Opposite sides of a parallelogram are equal.

(2) Opposite angles of a parallelogram are equal.

(3) Adjacent angles of a parallelogram are supplementary.



Proof: (1) ∠CAB = ∠ACB (alternate angles)

∠DAC = ∠BCA (alternate angles)

Hence ΔADC ≡ ΔCBA (one side and two angles).

∴ AB = DC and AD = BC.

(2) Clearly opposite angles are equal.

(3) If ∠A = ∠C = α and ∠B = ∠D = β then

$2\alpha + 2\beta = 360°$. Hence $\alpha + \beta = 180°$.

Theorem 6:

(1) The diagonals of a parallelogram bisect each other.

(2) The diagonals of a rectangle are equal in length.

(3) The diagonals of a rhombus are perpendicular.

Proof: The easiest proofs are to use vectors.

(1) The m**idpoints of the diagonals are** $\frac{\mathbf{0 + (u + v)}}{\mathbf{2}}$ **and** $\frac{\mathbf{u + v}}{\mathbf{2}}$, respectively and these coincide.

(2) If the parallelogram is a rectangle, $\mathbf{u}.\mathbf{v} = 0$.
Now $\mathbf{v} - \mathbf{u}$ is parallel, and equal, to the diagonal from $\mathbf{u}$ to $\mathbf{v}$. Its length is
$$|\mathbf{v} - \mathbf{u}|^2 = (\mathbf{v} - \mathbf{u}).(\mathbf{v} - \mathbf{u}) = |\mathbf{v}|^2 + |\mathbf{u}|^2 - 2\mathbf{u}.\mathbf{v} = |\mathbf{u}|^2 + |\mathbf{v}|^2.$$
The length of the other diagonal is
$$|\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}).(\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u}.\mathbf{v} = |\mathbf{u}|^2 + |\mathbf{v}|^2.$$
Hence the diagonals are equal in length.

(3) Suppose the parallelogram is a rhombus. Now $\mathbf{v} - \mathbf{u}$ is parallel, and equal, to the line segment joining $\mathbf{u}$ to $\mathbf{v}$.
Now $(\mathbf{v} - \mathbf{u}).(\mathbf{u} + \mathbf{v}) = |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$ since $|\mathbf{u}| = |\mathbf{v}|$.
Hence the diagonals are perpendicular.

§5.5. Varignon Parallelograms

Associated with any quadrangle there is a parallelogram, known as the **Varignon parallelogram**. It was discovered by Pierre Varignon (1654-1722).

The proof in *Geometry Revisited* by H.S.M. Coxeter and S.L. Greitzer uses areas. I prefer to avoid using areas where possible because of the difficulty in defining it rigorously. Besides, the proof has to be modified for non-simple or non-convex quadrilaterals. In any case, this is a case where the use of vectors is far simpler than normal geometric methods. Let the points be identified by vectors, using the same letters, so that A = **a**, etc.

Theorem 7 (Varignon): Let $\mathcal{P}$(ABCD) be a quadrangle and let P, Q, R, S be the midpoints of AB, BC, CD and DA respectively. Then ΔPQRS is a parallelogram.

Proof: Now **p** = ½(**a** + **b**), **q** = ½(**b** + **c**), **r** = ½(**c** + **d**) and **s** = ½(**d** + **a**). So PQ is represented by **q** – **p** = ½(**c** – **a**) and SR is represented by **r** – **s** = ½(**c** – **a**). Since these vectors are equal, PQ is parallel and equal to SR and so ΔPQRS is a parallelogram.

The Varignon parallelogram has some very special properties. I'll denote the parallelogram as ΔABCD and the Varignon parallelogram as ΔPQRS, where P, Q, R, S are the midpoints of AB, BC, CD and DA respectively.

Then, identifying these points as vectors we have:
p = ½(**a** + **b**), **q** = ½(**b** + **c**), **r** = ½(**c** + **d**), **s** = ½(**d** + **a**).

Theorem 8: Each pair of opposite sides of the Varignon parallelogram of a quadrangle are parallel to a diagonal of the quadrangle and has half its length.

Proof: PQ is represented by $\mathbf{q} - \mathbf{p} = ½(\mathbf{c} - \mathbf{a})$, which is parallel to the diagonal AC and has half its length. Similarly for PS.

Theorem 9: The perimeter of the Varignon parallelogram of a quadrangle is the sum of the lengths of its diagonals.
Proof: The perimeter of ΔABCD is $|\mathbf{c} - \mathbf{a}| + |\mathbf{d} - \mathbf{a}|$, which is AC + BD.

Theorem 10: The Varignon parallelogram of ΔABCD is a rhombus if and only if the diagonals are of equal length.
Proof: ΔPQRS is a rhombus if and only if

$$|\mathbf{p} - \mathbf{q}| = |\mathbf{p} - \mathbf{s}|,$$

that is, if and only if $½|\mathbf{c} - \mathbf{a}| = ½|\mathbf{b} - \mathbf{d}|$, which is ½AC = ½BD.

Theorem 11: The Varignon parallelogram of $\mathcal{P}$(ABCD) is a rectangle if and only if the diagonals are perpendicular.
Proof: $\mathcal{P}$(PQRS) is a rectangle if and only if

$$(\mathbf{p} - \mathbf{q}).(\mathbf{p} - \mathbf{s}) = 0,$$

that is, if and only if $(\mathbf{c} - \mathbf{a}).(\mathbf{b} - \mathbf{d}) = 0$, which is equivalent to AC ⊥ BD.

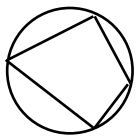
When it comes to the area of the Varignon parallelogram we revert to geometric methods. I'll denote the area of the polygon P by |P|.

Theorem 12: The Varignon parallelogram, 𝒫(PQRS), of 𝒫(ABCD) has half the area of the quadrangle itself.

Proof: |𝒫(PQRS)| = |𝒫(ABCD)| – |ΔPBQ| – |ΔRDS| – |ΔQCR| – |ΔSAP|
= |𝒫(ABCD)| – ¼|ΔABC| – ¼|ΔACD| – ¼|ΔBCD| – ¼|ΔABD|
= |𝒫(ABCD)| – ¼|𝒫(ABCD)| – ¼|𝒫(ABCD)|
= ½|𝒫(ABCD|.

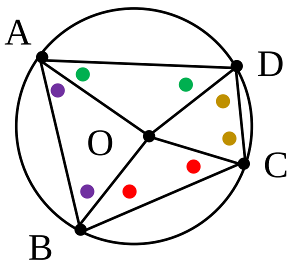
§5.6. Cyclic Quadrilaterals

If A, B, C, D lie on a circle, in cyclic order around the circle, the quadrangle ΔABCD is a quadrilateral, that is, it is convex and simple. It is called a **cyclic quadrilateral**.



Theorem 13: Opposite angles of a cyclic quadrilateral are supplementary.

Proof: Let ΔABCD be a quadrilateral where A, B, C, D lie on a circle with centre O.



Then OA = OB = OC = OD and hence the four triangles are isosceles.

Let $\alpha = \angle BAO = \angle OAB$,

$\beta = \angle CBO = \angle OCB$,

$\gamma = \angle DCO = \angle ODC$ and

$\delta = \angle DAO = \angle ODA$.

Then $2(\alpha + \beta + \gamma + \delta) = 360$ (angle sum of a quadrilateral).

Hence $(\alpha + \beta) = 180 - (\gamma + \delta)$ and

$(\alpha + \gamma) = 180 - (\beta + \delta)$,

From which it follows that opposite angles are supplementary.